Algebra II — Lesson 0825 - The Quadratic Formula

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The Quadratic Formula can be used to solve any quadratic equation. It is derived by completing the square of the standard form with variables as the coefficients.

Let us first review how to solve a quadratic equation whose leading coefficient is not one, with a couple of examples.

Example 1. Solve $2x^2 + 3x + 1 = 0$.

Solution. Complete the square in steps.

- $2x^2 + 3x + 1 = 0$ Step 0: put the equation in standard form.
- $x^2 + \frac{3}{2}x + \frac{1}{2} = 0$ Step 1: divide through by the leading coefficient.
- $x^2 + \frac{3}{2}x = -\frac{1}{2}$ Step 2: put the constant on the right hand side.
- $x^2 + \frac{3}{2}x + \frac{9}{16} = -\frac{1}{2} + \frac{9}{16}$ Step 3: add the thing that completes the square to both sides.
- $\left(x + \frac{3}{4}\right)^2 = \frac{1}{16}$ Step 4: factor the left and add the right.
- $x + \frac{3}{4} = \pm\frac{1}{4}$ Step 5: take the square root of both sides.
- $x = \frac{-3 \pm 1}{4}$ Step 6: put the constant on the right hand side.

Thus $x = -1$ or $x = -\frac{1}{2}$. The solution set is $\left\{-\frac{1}{2}, -1\right\}$.

Notice we could have solved this by factoring $2x^2 + 3x + 1 = (2x + 1)(x + 1) = 2\left(x + \frac{1}{2}\right)\left(x + 1\right)$. □

Example 2. Solve $5x^2 + 7x - 3 = 0$.

Solution. Complete the square in steps.

- $5x^2 + 7x - 3 = 0$ Step 0: put the equation in standard form.
- $x^2 + \frac{7}{5}x - \frac{3}{5} = 0$ Step 1: divide through by the leading coefficient.
- $x^2 + \frac{7}{5}x = \frac{3}{5}$ Step 2: put the constant on the right hand side.
- $x^2 + \frac{7}{5}x + \frac{49}{100} = \frac{3}{5} + \frac{49}{100}$ Step 3: add the thing that completes the square to both sides.
- $\left(x + \frac{7}{10}\right)^2 = \frac{109}{100}$ Step 4: factor the left and add the right.
- $x + \frac{7}{10} = \pm\frac{\sqrt{109}}{10}$ Step 5: take the square root of both sides.
- $x = \frac{-7 \pm \sqrt{109}}{10}$ Step 6: put the constant on the right hand side.

The solution set is $\left\{\frac{-7 \pm \sqrt{109}}{10}\right\}$. □

Proposition 1. *The solutions to the general quadratic equation*

$$ax^2 + bx + c = 0$$

are given by the quadratic formula*, which is*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solution. Complete the square in steps.

- $ax^2 + bx + c = 0$ — Step 0: put the equation in standard form.
- $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ — Step 1: divide through by the leading coefficient.
- $x^2 + \frac{b}{a}x = -\frac{c}{a}$ — Step 2: put the constant on the right hand side.
- $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ — Step 3: add the thing that completes the square to both sides.
- $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ — Step 4: factor the left and add the right.
- $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$ — Step 5: take the square root of both sides.
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Step 6: put the constant on the right hand side.

□

To use the quadratic formula, just identity the a, b, and c, and plug them in.

Example 3. Solve $4x^2 - 3x - 10 = x - 8 - x^2$.

Solution. First, put the equation in standard form. Add x^2 to both sides, subtract x from both sides, and add 8 to both sides to get

$$5x^2 - 4x - 2 = 0.$$

Here, $a = 5$, $b = -4$, and $c = -2$. Plug these in to get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 + 40}}{10} = \frac{2 \pm \sqrt{14}}{5}.$$

□

Example 4. Find two numbers whose sum is 7 and whose product is 11.

Solution. Let x and y be these numbers. We have $x + y = 7$ and $xy = 11$. Then $y = 7 - x$, and plugging this into the second equation gives $x(7 - x) = 11$, so $7x - x^2 = 11$. Put this in standard form to get $x^2 - 7x + 11 = 0$. Thus, the quadratic formula gives

$$x = \frac{7 \pm \sqrt{49 - 44}}{2} = \frac{7 \pm \sqrt{5}}{2}.$$

If we let $x = \frac{7 + \sqrt{5}}{2}$, then $y = \frac{7 - \sqrt{5}}{2}$.

Let's check this with the formula $(a + b)(a - b) = a^2 - b^2$. We have

$$xy = \frac{7 + \sqrt{5}}{2} \cdot \frac{7 - \sqrt{5}}{2} = \frac{(7 + \sqrt{5})(7 - \sqrt{5})}{4} = \frac{49 - 5}{4} = \frac{44}{4} = 11.$$

Moreover,

$$x + y = \frac{7 + \sqrt{5}}{2} + \frac{7 - \sqrt{5}}{2} = \frac{14}{2} = 7.$$

□